Mystery Plots: Motivating Algebraic Model Building with Dynamic Sketches

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`"No one ever washes a rental car." - Unknown`

1 Introduction

Functions are central to the study of mathematics at the secondary level. As Froelich, Bartkovich, and Foerster (1993) note, "the concept of function is probably the most important idea in mathematics" (p. 1). Although students spend significant time working with functions at the secondary level, much of this time is spent transforming familiar "parent" functions - for instance, stretching, reflecting, and translating exponential, quadratic, square root, and sinusoidal functions - rather than creating *original* functions. The tendency to modify and "borrow" rather than create impacts students' attitudes regarding mathematics. Functions become "gifts" from teachers (or the back of the textbook) rather than objects of discovery in their own right. Mathematics is not construed as a creative area of study.

In this paper, we explore the use of dynamic geometry software (DGS) as a medium for changing student and teacher interactions (and attitudes) with functions. We offer three examples of sketches that may be used to encourage students to build their own functions. Moreover, we share a strategy for developing additional sketches, namely our three-step MTA process (Measure - Trace - Algebratize). Note that these steps roughly correspond to concrete, iconic, and symbolic levels of representation proposed by Bruner (1960; 1966). As our examples illustrate, the MTA approach provides students with opportunities to explore and construct remarkably non-standard functions - often beautiful, unexpected, and thoroughly original.

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2 Introducting the MTA Process

Consider the three steps of the MTA process in more detail.

1. <u>Measure</u>. Constructing a familiar object (e.g., rectangle, circle, triangle) and look for relationships among measures (e.g., relationships between angles, side lengths, areas).

2. <u>Trace</u>. Next, plot an ordered pair of two seemingly related measures (e.g., plot (area,perimeter) of a triangle). Examine a trace as points are dragged in the sketch. In particular, consider the domain and range of the trace. Specifically, look for traces that suggest functions.

3. <u>Algebratize</u>. Lastly, describe relationships suggested by tracings symbolically by means of one (or more) algebraic functions (i.e., function models).

It is worthwhile to note that both teachers and students may engage in the MTA process. We use the technique to build tasks for our students. Likewise, students may use the process to construct their own functions.

2.1 Mystery Plot 1 (MP1)

We used the MTA approach to create the following task, referred to as Mystery Plot 1 (MP1).

> Two vertices of a right triangle are fixed at $B(0,0)$ and $C(6,0)$. Vertex $D(6,y)$ may move anywhere along the line $x = 6$. Determine the function $f(P) = A$ that describes A, the area of $\triangle BCD$, with respect to P, the triangle's perimeter.

We created the Mystery Plot 1 (MP1) task by completing the first two steps of the MTA approach. First, we created a "generic" right triangle with base along the x-axis. Then we analyzed various measurements (area, perimeter, side lengths, slope of sides, etc.), looking for possible relationships. After noting a possible relationship among these measures, we constructed the ordered pair $Q = (perimeter, area)$, tracing this point as we dragged vertex D. Initially we were somewhat surprised to find that the plot appeared to be linear (as shown in Figure 1).
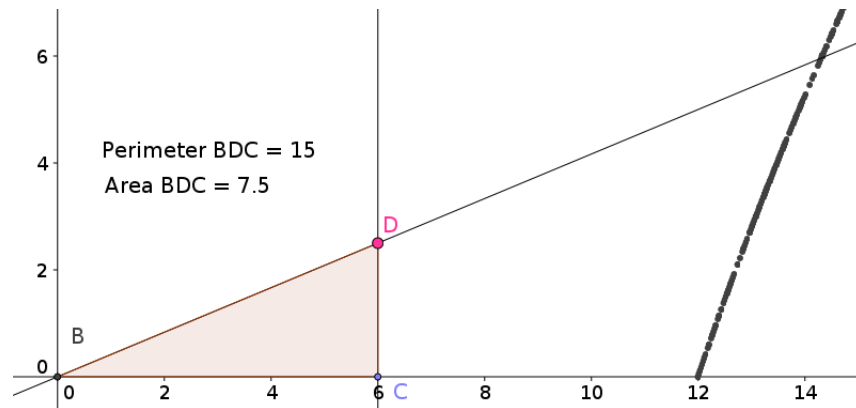
Figure 1: *Dynamic sketch of mystery plot (MP1) in* GeoGebra *DGS.*

In MP1, we ask students to complete Step 3 of the MTA approach (*Algebratize*). Initially, when students attempt to describe the trace of the ordered pair (*perimeter, area*) symbolically, they are convinced that the relationship between Perimeter and Area is linear. This conjecture is based wholly on visual evidence provided by graph rather than on careful consideration of the relationship between variables.

Noting that ordered pairs $(12, 0)$ and $(15, 7.5)$ are plotted by dragging point D, students follow their initial conjecture and construct a linear function to describe the plot. Such work is highlighted in Figure 2.

Figure 2: *Algebraic work for possible function for area in terms of perimeter.*

When the linear function from Figure 2 is superimposed on the plot of various perimeter-area pairs, results are not wholly unsatisfactory, although the curvature of the original data points becomes more apparent. While many falsely believe that $f(P) = 2.5P - 30$

describes the relationship between perimeter and area, other students become skeptical of this conclusion. Differences of opinion fuel further investigation as we encourage students to analyze underlying geometric and algebraic relationships between area and perimeter rather than basing their findings wholly on visual perception.

Letting $h = CD$, students use the Pythagorean Theorem to determine the length of BD in terms of h, as shown in Figure 3.

$$(BC)^2 + (CD)^2 = (BD)^2$$
$$6^2 + h^2 = (BD)^2$$
$$36 + h^2 = (BD)^2$$
$$\sqrt{36 + h^2} = BD$$

Figure 3: *Calculating BD in terms of h.*

From this, students find a formula for perimeter, P, in terms of h. This approach is highlighted in Figure 4 (left). Similarly, students use the familiar area formula for a triangle to find a formula for area, A, in terms of h, as shown in Figure 4 (right).

$$P = BC + CD + BD$$
$$= 6 + h + \sqrt{36 + h^2}$$

$$A = \frac{1}{2} \cdot base \cdot height$$
$$= \frac{1}{2} \cdot 6 \cdot h = 3h$$

Figure 4: *(Left) Calculating perimeter; (Right) Calculating area.*

Combining the results in Figure 4, we see that $P = 6 + \frac{A}{3} + \sqrt{36 - \left(\frac{A}{3}\right)^2}$. As Figure 5 illustrates, this relationship may be used to find a function for A in terms of P.

$$(P-6) - \frac{A}{3} = \sqrt{36 + \left(\frac{A}{3}\right)^2}$$

$$\left[(P-6) - \frac{A}{3}\right]^2 = 36 + \left(\frac{A}{3}\right)^2$$

$$\left((P-6) - \frac{A}{3}\right)\left((P-6) - \frac{A}{3}\right) = 36 + \left(\frac{A}{3}\right)^2$$

$$(P-6)^2 - \frac{2}{3}A(P-6) + \left(\frac{A}{3}\right)^2 = 36 + \left(\frac{A}{3}\right)^2$$

$$-\frac{2}{3}(P-6)A = 36 - (P-6)^2$$

$$A = -3\left(\frac{36 - (P-6)^2}{P-6}\right)$$

Figure 5: *Calculating area, A, in terms of perimeter, P.*

As Figure 6 illustrates, the function derived using algebraic and geometric relationships between area and perimeter provides a better "fit" than the linear function. For the function $P = 6 + \frac{A}{3} + \sqrt{36 - \frac{A^2}{3}}$, we use domain $P > 12$, reflecting mathematical content from which the functions were derived (the smallest value of the perimeter).
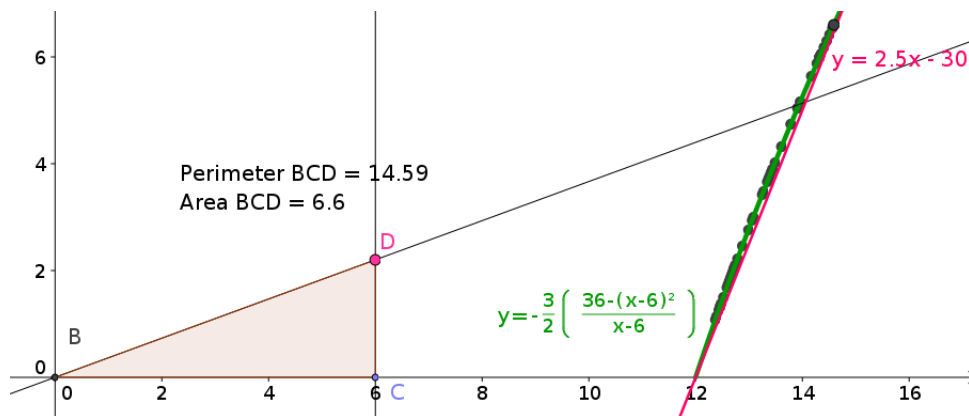

Figure 6: *Comparing proposed linear and non-linear area models.*

The mathematics content required to construct an accurate function model for the MP1 task is noteworthy for several reasons. First, the use of DGS did not replace the need for students to think deeply about mathematics. Indeed, "incorrect" conjectures encouraged

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by the sketch prompted students to challenge their initial intuitions. Secondly, the task requires students to connect algebra and geometry content creatively. Recognizing that the Pythagorean Theorem can be used to determine the perimeter of BCD and that the resulting expression can be expressed in terms of the area of the triangle are essential steps for solving the task. Lastly, although the algebraic work involved in the task requires little (if any) content beyond second year algebra, the resulting function is not one typically studied in a second year algebra course. In this sense, the task requires students to engage in "outside the box" thinking about functions.

2.2 A Second Mystery Plot (MP2)

We used the MTA approach yet again to create the following task, referred to as Mystery Plot 2 (MP2).

> Two vertices of a triangle are fixed at $A(0,0)$ and $B(-2,0)$. Vertex C may move anywhere in the coordinate plane. The plot of the area of $\triangle ABC$ with respect to the triangle's perimeter is bounded above by function $f(x)$. Determine $f(x)$.

As before, we created the Mystery Plot 2 task by completing the first two steps of the MTA approach. First, we constructed a "generic" triangle and analyzed various measurements. Secondly, we plotted the triangle's area with respect to perimeter and dragged a vertex. We were genuinely surprised to find that the plot appeared to be bounded above by a mystery curve (as shown in Figure 7). We ask our students to complete Step 3 of the approach (*Algebratize*) by constructing a symbolic representation of the bounding curve, $f(x)$. One method for algebratizing $f(x)$ makes use of the fact that *isosceles* triangle ABC has maximal area for a given perimeter. Hence, when a student positions point C along the perpendicular bisector of $\overline{AB}$, the plot of the triangle's area with respect to perimeter (i.e., Q) lies on $f(x)$. With this observation in mind, constructing bounding function $f(x)$ proceeds in the following manner.

Given $AB = 2$, the area of $\triangle ABC$ is numerically equal to the height of the triangle, as measured from $\overline{AB}$. By the Pythagorean Theorem, the height of $\triangle ABC$, and thereby the area, is given by the expression

$$Area(\triangle ABC) = \sqrt{(AC)^2 - 1^2}. \tag{1}$$

Our desire is to express the area in terms of the perimeter. Given $\triangle ABC$ being isosceles with $AC = AB$ and $AB = 2$, we can express the perimeter in terms of AC.
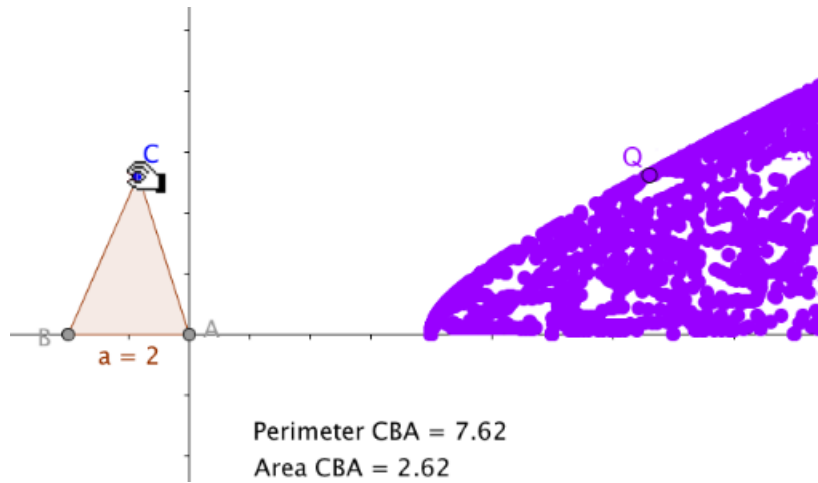
$$P = AB + AC + CB = 2 + 2AC \tag{2}$$

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Figure 7: *Dynamic sketch of mystery plot (MP2) in* GeoGebra *DGS.*

Solving (2) for AC, we have

$$AC = \frac{P-2}{2} = 0.5P - 1 \tag{3}$$

Substituting $0.5P - 1$ for AC in (1), we have

$$Area(\triangle ABC) = \sqrt{(0.5P - 1)^2 - 1^2} \tag{4}$$

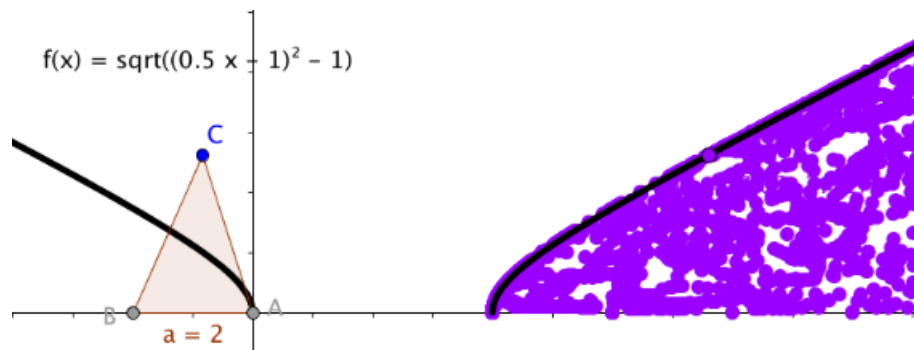Plotting $f(x) = \sqrt{(0.5P - 1)^2 - 1^2}$ yields the graph shown in Figure 8.

Figure 8: *Graph of $f(x)$ superimposed on original sketch.*

Note that since P is squared, negative P values generate outputs. Hence, the domain of the function must be restricted to values of $P > 0$.

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The mathematics content required to construct a function model for the MP2 task shares features of MP1 work. First, the resulting function is not one typically found in school texts. Secondly, algebraic manipulation plays an essential role in the construction of the bounding function. Thirdly, the task requires students to connect algebra and geometry content creatively. Recognizing that isosceles $\triangle ABC$ generates points on boundary function $f(x)$ is essential for solving the task. This result may not be obvious to all students. For these students, this intermediate result should be considered as a sub-problem and may require teachers' special attention

2.3   A Third Mystery Plot (MP3)

To construct our third mystery plot, we begin (once again) with a triangle. This time, however, we construct a centroid of the triangle and examine measures of angles formed in the construction as we drag *any* of the triangle vertices. Consider, for instance, $\triangle ABC$ with centroid G as depicted in Figure 9.
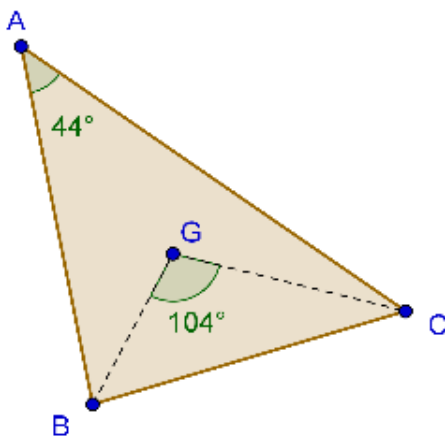

Figure 9: *Scalene $\triangle ABC$ with centroid G and $\angle BGC$.*

Upon measuring $\angle BAC$ and $\angle BGC$, we noticed an apparent relationship between the two angles as vertices were dragged. Before we trace the ordered pair $m\angle BAC$, $m\angle BGC$ with our students, we encourage them to build conjectures concerning the nature of the plot. Consider the following talking points.

- Certainly, as the measure of $\angle BAC$ becomes large, it is intuitively clear that the measure of $\angle BGC$ will also become large.

- Indeed, if we were to graph the measure of $\angle BGC$ with respect to the measure of $\angle BAC$, we would conjecture that the point $(180, 180)$ would be on the graph.

- Likewise, it is also intuitively clear that as the $m\angle BAC$ becomes quite small, that the measure of $\angle BGC$ will also become quite small. Students may visualize $\angle BAC$ getting smaller as points B and C are dragged closer together, thereby making $\angle BGC$ just as small.

Through discussions of this sort, many students correctly conjecture that $(0,0)$ will also be on the graph. What happens in between $(0,0)$ and $(180,180)$ is still left to be discovered, but for many it is intuitively clear that these two points would be connect by some sort of a smooth graph. After all, the graphs of the majority of functions the students encounter are characterized as smooth. The plot shown in Figure 10, created by dragging **all of the triangle vertices**, one after another, is not what most students expect - provided we have had the kind of discussion with students as described above.
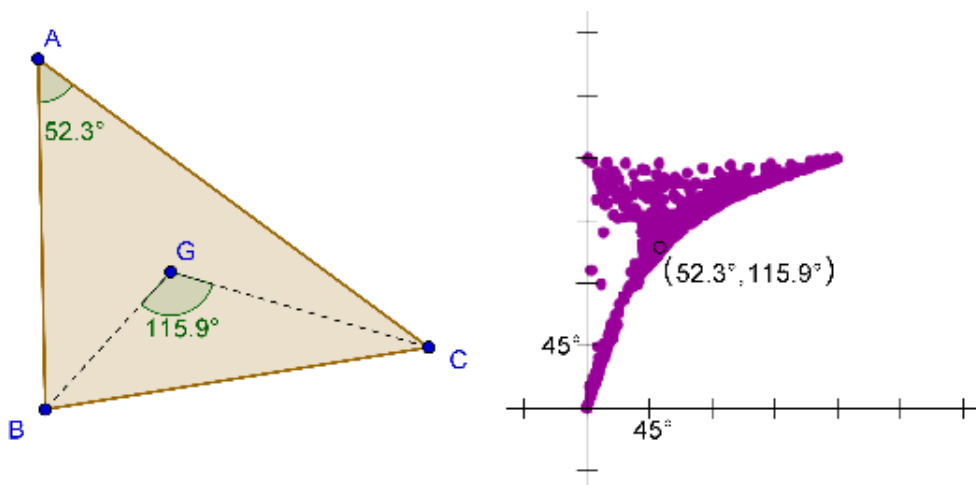

Figure 10: *Plot of $m(\angle BGC)$ with respect to $m(\angle BAC)$.*

This rather surprising results leads us to our third (and final) Mystery Plot.

Mystery Plot (MP3). Construct the centroid G of $\triangle ABC$. The plot of $\angle BGC$ with respect to $\angle BAC$ is bounded below by the function $f(x)$. Determine $f(x)$.

Trying to understand the plot in Figure 10 by randomly dragging the vertices is not an efficient method for revealing its underlying mysteries. In this instance, random dragging generates a plot with large "gaps." Purposeful dragging, on the other hand, enables one to "fill in" the the feasible region with all possible ordered pairs. Thoughtful manipulation of vertex A encourages students to discover an important feature of the plot - namely, that different measures of $\angle BGC$ are produced by the same measures of $\angle BAC$. This is illustrated in Figure 11. As was the case in MP2, our third task generates a multi-valued function.
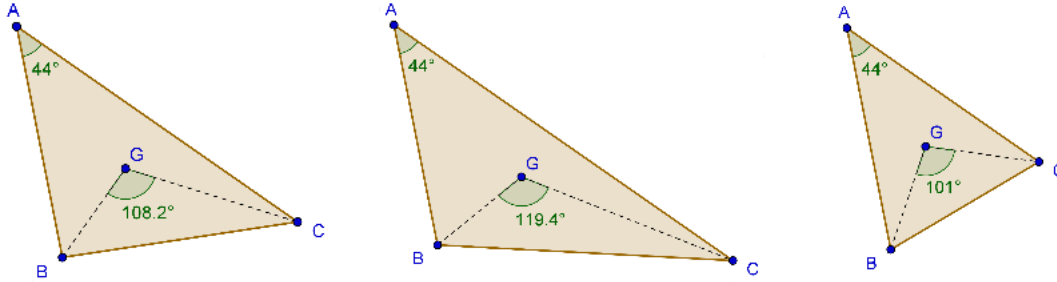
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Figure 11: *Different measures of ∠BGC are produced by the same measure of ∠BAC.*

Dragging in such a manner as to fill the entire feasible region requires students to make connections between the angles in the triangle and the ordered pairs on the graph. Knowing what ordered pairs one wants to generate is a far cry from knowing how to change the shape of the triangle to make the desired result happen. When students drag to fill the plot of the feasible region in the manner suggested in Figure 12, they may notice that the region appears symmetric with respect to the line $y = -x + 180$. Based on this observation, many conjecture that the desired boundary curve is an arc of a circle, or perhaps part of a hyperbola (refer to Figure 12).
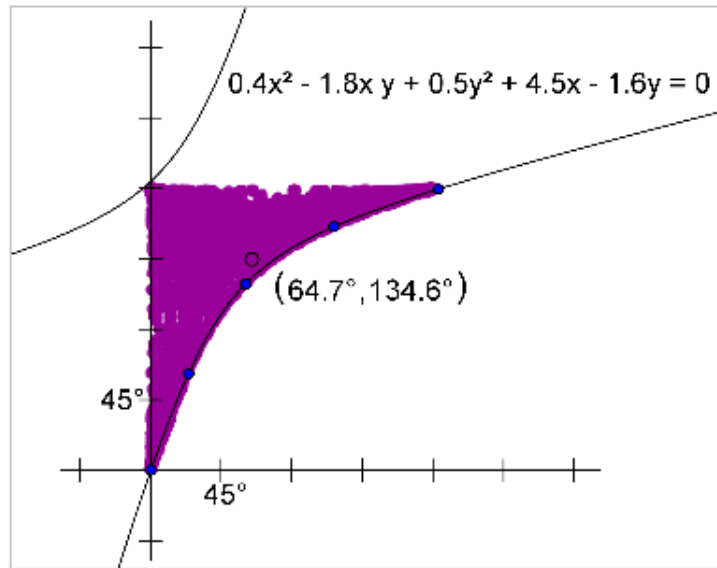

$$0.4x^2 - 1.8xy + 0.5y^2 + 4.5x - 1.6y = 0$$

$$(64.7°, 134.6°)$$

Figure 12: *The function $f(x)$ appears to be modeled by a hyperbola.*

In discovering the nature of $f(x)$ through purposeful dragging, it appears that the boundary curve is generated by dragging point A along the perpendicular bisector of $\overline{BC}$ (Figure 13).

Accepting this assumption, it appears that this Mystery Plot has something in common with Mystery Plot 2 - namely, that a solution may follow from knowledge of isosceles triangles.
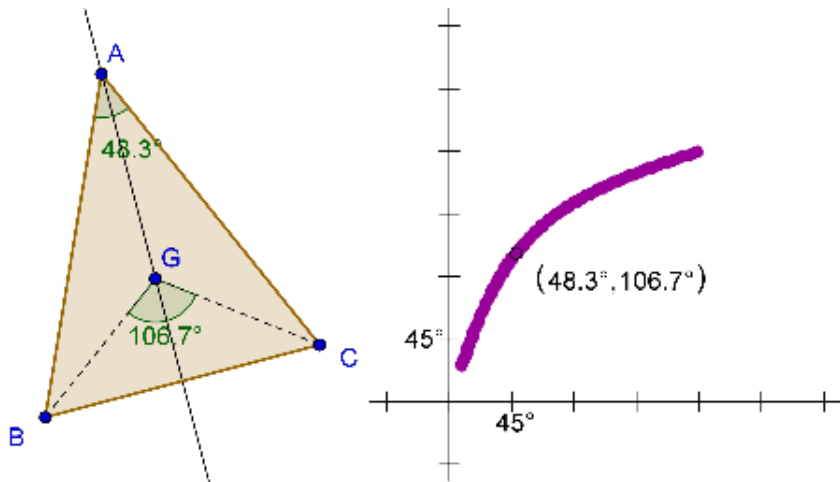

Figure 13: *The function $f(x)$ appears to be traced when A is dragged along the perpendicular bisector of $\triangle ABC$.*

Working from this isosceles triangle assumption, and drawing on our knowledge of centroids and medians, students can begin to build a function describing the lower boundary of the plot region.
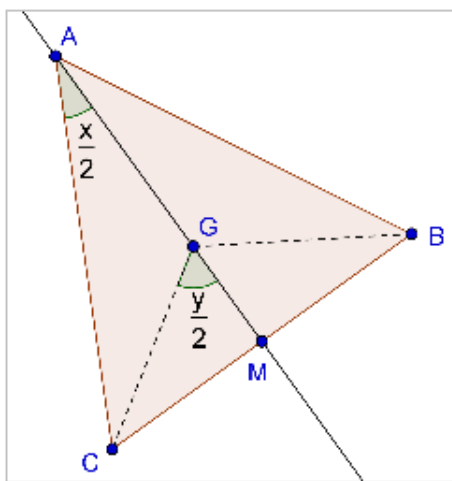

Figure 14: *Consider angles formed by perpendicular bisector $\overline{AM}$ of $\triangle ABC$.*

Focusing our attention on $\triangle ACM$ and $\triangle GCM$ with $\angle CAM$ and $\angle CGM$ labeled as shown in Figure 14. These two triangles, besides both being right triangles (the median drawn

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from the vertex angle in an isosceles triangle is also an altitude), share several other significant features. First, they obviously share side $\overline{CM}$. Less obviously, side $\overline{AM}$ is three times the length of $\overline{GM}$ (the centroid divides the median in a 1:2 ratio). With respect to $\angle CAM$ and $\angle CGM$, the opposite side and the adjacent side in each triangle are related. Hence, we will look to the tangent ratio.

In $\triangle ACM$ we have

$$tan\left(\frac{x}{2}\right) = \frac{CM}{AM}. \tag{5}$$

In $\triangle CGM$ we have

$$tan\left(\frac{y}{2}\right) = \frac{CM}{GM}. \tag{6}$$

Solving (5) and (6) for CM and using the fact that $AM = 3GM$, we have

$$GM \cdot tan\left(\frac{y}{2}\right) = 3 \cdot GM \cdot tan\left(\frac{x}{2}\right). \tag{7}$$

Solving (7) for y, we have

$$y = 2 \cdot tan^{-1}\left(3 \cdot tan\left(\frac{x}{2}\right)\right). \tag{8}$$

Graphing the function, we see it appears to match the boundary (refer to Figure 15). Of course, the graph of this function has ordered pairs in the third quadrant, but for our purposes, we will only consider that portion of the graph that is in the domain $0 < x < 180$.

Note that we have not conclusively proved that the function in Figure 15 is "perfect" model for the plotted points. In fact, this provides students with a rich opportunity to explore the fit of various functions.

3 Conclusion

In this paper, we've proposed a technique for constructing function tasks for students using dynamic geometry software (DGS). Using the three-step MTA approach (Measure-Trace-Algebratize), teachers construct environments for students that require active exploration of function through purposeful dragging. As the previous three examples have illustrated, DGS may be used to enable students to construct their own functions as producers - rather than consumers - of mathematics. As students construct a function models for traced data,
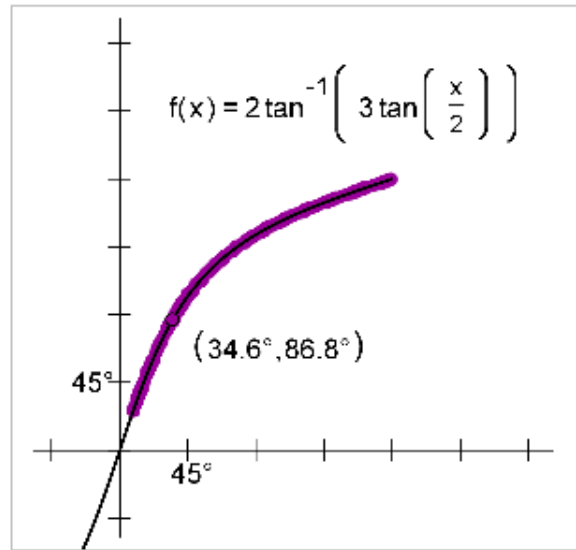
Figure 15: *Plotted data with proposed function model.*

they connect basic geometry and algebra in ways not commonly encountered in school texts. Algebraic manipulation plays an essential role in the construction of bounding functions.

By honoring students' thinking through mathematical exploration, experimentation, and conjecturing, we encourage students to "own" the mathematics that they experience in school classrooms. Using DGS, we encourage students to make "incorrect" conjectures based on faulty intuition. Once students "buy in" to their initial hypotheses, we challenge their intuitions - creating cognitive dissonance that encourages deep learning - then and "hook" them into rich geometric and algebraic explorations.

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